SELF-AVERAGING SCALING LIMITS OF TWO-FREQUENCY WIGNER DISTRIBUTION FOR RANDOM PARAXIAL WAVES

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Abstract. Two-frequency Wigner distribution is introduced to capture the asymptotic behavior of the space-frequency correlation of paraxial waves in the radiative transfer limits. The scaling limits give rises to deterministic transport-like equations. Depending on the ratio of the wavelength to the correlation length the limiting equation is either a Boltzmann-like integral equation or a Fokker-Planck-like differential equation in the phase space. The solutions to these equations have a probabilistic representation which can be simulated by Monte Carlo method. When the medium fluctuates more rapidly in the longitudinal direction, the corresponding Fokker-Planck-like equation can be solved exactly.

1. Introduction

A central quantity of wave propagation in random media is the correlation of wave field at two space-time points. Through spectral decomposition of the time-dependent signal the space-time correlation is equivalent to the space-frequency correlation of the wave field [16]. The main focus of the present work is to derive rigorously closed form equations governing the space-frequency correlation of paraxial waves in the radiative transfer regime.

For optical wave propagating through the turbulent atmosphere, the complex-valued wave amplitude is governed by the stochastic Schrödinger (paraxial) equation with a white-noise potential [17]. The conventional approach uses the two-frequency mutual coherence function and various ad hoc approximations [13]. Recently, we introduced the two-frequency Wigner distribution in terms of which we derived rigorously a complete set of two-frequency all-order moment equations and solved exactly the mutual coherence function in the geometrical optics regime [7].

In this paper, we consider the different regime of radiative transfer and prove the *self-averaging* convergence of the two-frequency Wigner distribution for the paraxial wave equation. In other words, in radiative transfer, the whole hierarchy of two-frequency moment equations is reduced to a single radiative-transfer-like equation.

Let L_z and L_x be, respectively, the characteristic macroscopic length scales of the wave beam in the longitudinal and transverse directions. We assume the phase speed in the vacuum is unity. The Fresnel number commonly defined as

$$F = \frac{L_x^2}{\lambda_0 L_z}$$

with the central wavelength $\lambda_0 = 2\pi/k_0$ measures the significance of Fresnel diffraction. In all the scalings considered here the corresponding Fresnel number is large, indicating strong Fresnel diffraction effect.

We use λ_0, L_z, L_x to non-dimensionalize the paraxial wave equation [13]. Let k_1, k_2 be two (relative) wavenumbers nondimensionalized by the central wavenumber k_0. Then the wave field Ψ_j

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of wavenumber k_j satisfies

$$i\frac{\partial}{\partial z}\Psi_j(z,\mathbf{x})+\frac{\gamma}{2k_j}\nabla^2_{\mathbf{x}}\Psi_j(z,\mathbf{x})+\frac{\mu k_j}{\gamma}V(\frac{zL_z}{\ell_z},\frac{\mathbf{x}L_x}{\ell_x})\Psi_j(z,\mathbf{x})=0,\quad j=1,2 \tag{1}$$

where γ is the reciprocal of Fresnel number, V represents the refractive index fluctuation with the correlation lengths ℓ_z and ℓ_x in the longitudinal and transverse directions, respectively, and μ is the magnitude of the fluctuation. We denote the ratios ℓ_z/L_z and ℓ_x/L_x, respectively, by ρ_z and ρ_x which then are the correlation lengths of the medium fluctuations in the unit of characteristic length scales of the wave beam.

Let us now describe a family of scaling limits which would yield non-trivial self-averaging limit for the space-frequency correlation of the wave beam. The main characteristic of the scaling limits is that the transverse correlation length ℓ_x is much smaller than the beam width L_x but is comparable or much larger than the central wavelength λ_0. Roughly speaking, the wave beam experiences the spatial diversity as it propagates through the medium but it does not do so over a single wavelength in order to avoid the effective medium regime. This condition is best described by dimensionless quantities as

$$\theta\equiv\gamma/\rho_x=O(1) \tag{2}$$

which includes the possibility that $\theta\ll 1$. This is the radiative transfer regime for monochromatic waves [8].

To reduce the complexity of technique, however, we restrict ourselves mainly to the longitudinal case where $\rho_z/\rho_x=O(1)$, including the possible scenario $\rho_z/\rho_x\ll 1$. In this case, the medium fluctuation of the following order of magnitude

$$\mu=\frac{\gamma}{\theta\sqrt{\rho_z}}=\frac{\rho_x}{\sqrt{\rho_z}} \tag{3}$$

would lead to non-trivial multiple scattering effect (see [6], [8] and references therein).

For wave fields of two different frequencies to interfere coherently, the unscaled frequency shift needs to be limited to $O(1)$ regardless the central frequency. In other words, we look for $O(1)$ coherence bandwidth which is a characteristic of the random medium. Again, this is best expressed in the dimensionless quantities as

$$\lim_{\ell_x\to 0}k_1=\lim_{\ell_x\to 0}k_2=k,\quad \lim_{\ell_x\to 0}\gamma^{-1}k^{-1}(k_2-k_1)=\beta. \tag{4}$$

We assume $\beta>0$ below. We call this the two-frequency radiative transfer scaling limits.

Let us now review the basic framework of two-frequency Wigner distribution in the paraxial regime. The two-frequency Wigner distribution is defined as [7]

$$W(z,\mathbf{x},\mathbf{p})=\frac{1}{(2\pi)^d}\int e^{-i\mathbf{p}\cdot\mathbf{y}}\Psi_1(z,\frac{\mathbf{x}}{\sqrt{k_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{k_1}})\Psi_2^*(z,\frac{\mathbf{x}}{\sqrt{k_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{k_2}})d\mathbf{y} \tag{5}$$

where the scaling factor $\sqrt{k_j}$ is introduced so that W satisfies a closed-form equation (see below). Note here the scaling factor for the parabolic wave is different from that for the spherical wave introduced in [10].

The following property can be derived easily from the definition

$$\|W(z,\cdot,\cdot)\|_2 \;=\; \left(\frac{\sqrt{k_1k_2}}{2\gamma\pi}\right)^{d/2}\|\Psi_1(z,\cdot)\|_2\|\Psi_2(z,\cdot)\|_2.$$

Since Ψ_j are governed by the Schrödinger-like equation the L^2-norm does not change with z, i.e., $\|W(z,\cdot,\cdot)\|_2 = \|W(0,\cdot,\cdot)\|_2$. The Wigner distribution has the following obvious properties:

$$\int W(z,\mathbf{x},\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{y}}d\mathbf{p} = \Psi_1(z,\frac{\mathbf{x}}{\sqrt{k_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{k_1}})\Psi_2^*(z,\frac{\mathbf{x}}{\sqrt{k_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{k_2}}) \tag{6}$$

$$\int_{\mathbb{R}^d} W(z,\mathbf{x},\mathbf{p})e^{-i\mathbf{x}\cdot\mathbf{q}}d\mathbf{x} = \left(\frac{\pi^2\sqrt{k_1k_2}}{\gamma}\right)^d \widehat{\Psi}_1(z,\frac{\mathbf{p}\sqrt{k_1}}{4\gamma}+\frac{\sqrt{k_1}\mathbf{q}}{2})\widehat{\Psi}_2^*(z,\frac{\mathbf{p}\sqrt{k_2}}{4\gamma}-\frac{\sqrt{k_2}\mathbf{q}}{2}) \tag{7}$$

and so contains essentially all the information in the two-point two-frequency function.

The Wigner distribution W_z satisfies the Wigner-Moyal equation exactly [7]

$$\frac{\partial W}{\partial z}+\mathbf{p}\cdot\nabla_{\mathbf{x}}W+\frac{1}{\sqrt{\rho_z}}\mathcal{L}_zW=0 \tag{8}$$

where the operator $\mathcal{L}_z$ is formally given as

$$\mathcal{L}_zW = i\int \theta^{-1}\left[e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1W(z,\mathbf{x},\mathbf{p}+\frac{\theta\mathbf{q}}{2\sqrt{k_1}})-e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2W(z,\mathbf{x},\mathbf{p}-\frac{\theta\mathbf{q}}{2\sqrt{k_2}})\right]\widehat{V}(\frac{z}{\rho_z},d\mathbf{q})$$

with $\tilde{\mathbf{x}}=\mathbf{x}/\rho_x$ being the 'fast' transverse variable. As $\rho_z\to 0$, $\rho_z^{-1/2}\mathcal{L}_z$ displays the classical central limit scaling in the z-variable and thus this is referred to as the longitudinal case. The complex conjugate $W_z^{\varepsilon *}(\mathbf{x},\mathbf{p})$ satisfies a similar equation

$$\frac{\partial W^*}{\partial z}+\mathbf{p}\cdot\nabla_{\mathbf{x}}W^*+\frac{1}{\sqrt{\rho_z}}\mathcal{L}_z^*W^*=0 \tag{9}$$

where

$$\mathcal{L}_z^*W^* = i\int \theta^{-1}\left[e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2W^*(z,\mathbf{x},\mathbf{p}+\frac{\theta\mathbf{q}}{2\sqrt{k_2}})-e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1W^*(z,\mathbf{x},\mathbf{p}-\frac{\theta\mathbf{q}}{2\sqrt{k_1}})\right]\widehat{V}(\frac{z}{\rho_z},d\mathbf{q}).$$

We use the following definition of the Fourier transform and inversion:

$$\mathcal{F}f(\mathbf{p}) = \frac{1}{(2\pi)^d}\int e^{-i\mathbf{x}\cdot\mathbf{p}}f(\mathbf{x})d\mathbf{x}$$
$$\mathcal{F}^{-1}g(\mathbf{x}) = \int e^{i\mathbf{p}\cdot\mathbf{x}}g(\mathbf{p})d\mathbf{p}.$$

When making a *partial* (inverse) Fourier transform on a phase-space function we will write $\mathcal{F}_1$ (resp. $\mathcal{F}_1^{-1}$) and $\mathcal{F}_2$ (resp. $\mathcal{F}_2^{-1}$) to denote the (resp. inverse) transform w.r.t. $\mathbf{x}$ and $\mathbf{p}$ respectively.

2. Formulation and theorems

To describe the scaling limits more efficiently we introduce two controlling parameters $\varepsilon,\alpha>0$ and set

$$\rho_z=\varepsilon^2,\quad \rho_x=\varepsilon^{2\alpha}. \tag{10}$$

By (2)-(4) all the other parameters can be expressed in terms of ε, α and θ. The radiative transfer scalings in the longitudinal case then correspond to

$$\varepsilon\to 0,\quad \alpha\in(0,1],\quad \lim_{\varepsilon\to 0}\theta<\infty \tag{11}$$

along with (2)-(4).

To get the self-averaging result, it is essential to consider the weak formulation of the Wigner-Moyal equation: To find W_z^ε in the space $C([0,\infty);L_w^2(\mathbb{R}^{2d})$ of z-continuous, $L^2(\mathbb{R}^{2d})$-valued processes such that $\|W_z^\varepsilon\|_2\le\|W_0\|_2,\forall z>0$, and

$$\langle W_z^\varepsilon,\Theta\rangle-\langle W_0,\Theta\rangle = \int_0^z\langle W_s^\varepsilon,\mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta\rangle\,ds+\frac{1}{\varepsilon}\int_0^z\langle W_s^\varepsilon,\mathcal{L}_s^*\Theta\rangle\,ds, \tag{12}$$

where $W_0 \in L^2(\mathbb{R}^{2d})$ is the initial condition and we consider all the smooth test functions of compact supports $\Theta \in C_c^\infty(\mathbb{R}^{2d})$. Here and below $L^2_w(\mathbb{R}^{2d})$ is the space of complex-valued square integrable functions on the phase space $\mathbb{R}^{2d}$ endowed with the weak topology and the inner product

$$\langle W_1, W_2 \rangle = \int W_1^*(\mathbf{x},\mathbf{p}) W_2(\mathbf{x},\mathbf{p}) d\mathbf{x} d\mathbf{p}.$$

We define for every realization of V_z^ε the operator $\mathcal{L}_z^*$ to act on a phase-space test function Θ as

$$\mathcal{L}_z^* \Theta(\mathbf{x},\mathbf{p}) \equiv -i\theta^{-1} \mathcal{F}_2 \left[\delta_\varepsilon V_z^\varepsilon(\mathbf{x},\mathbf{y}) \mathcal{F}_2^{-1} \Theta(\mathbf{x},\mathbf{y}) \right] \tag{13}$$

with the difference operator δ_ε given by

$$\delta_\varepsilon V_z^\varepsilon(\mathbf{x},\mathbf{y}) \equiv k_1 V_z^\varepsilon\left(\frac{\mathbf{x}}{\sqrt{k_1}} + \frac{\theta\mathbf{y}}{2\sqrt{k_1}}\right) - k_2 V_z^\varepsilon\left(\frac{\mathbf{x}}{\sqrt{k_2}} - \frac{\theta\mathbf{y}}{2\sqrt{k_2}}\right) \tag{14}$$

We define $\mathcal{L}_z$ in the similar way.

The existence of solutions in the space $C([0,\infty); L^2_w(\mathbb{R}^{2d}))$ can be established by the same weak compactness argument as in [7]. We will not, however, address the uniqueness of solution for the Wigner-Moyal equation (12) but we will show that as $\varepsilon \to 0$ any sequence of weak solutions to eq. (12) converges in a suitable sense to the unique solution of a deterministic equation (see Theorem 1 and 2).

We assume that $V_z(\mathbf{x}) = V(z,\mathbf{x})$ is a real-valued, centered, z-stationary, $\mathbf{x}$-homogeneous ergodic random field admitting the spectral representation

$$V_z(\mathbf{x}) = \int \exp(i\mathbf{p}\cdot\mathbf{x}) \hat{V}_z(d\mathbf{p})$$

with the z-stationary spectral measure $\hat{V}_z(\cdot)$ satisfying

$$\mathbb{E}[\hat{V}_z(d\mathbf{p})\hat{V}_z(d\mathbf{q})] = \delta(\mathbf{p}+\mathbf{q})\Phi_0(\mathbf{p}) d\mathbf{p} d\mathbf{q}.$$

The transverse power spectrum density is related to the full power spectrum density $\Phi(w,\mathbf{p})$ as $\Phi_0(\mathbf{p}) = \int \Phi(w,\mathbf{p}) dw$. The power spectral density $\Phi(\vec{\mathbf{k}})$ satisfies $\Phi(\vec{\mathbf{k}}) = \Phi(-\vec{\mathbf{k}}), \forall \vec{\mathbf{k}} = (w,\mathbf{p}) \in \mathbb{R}^{d+1}$ because the electric susceptibility field is assumed to be real-valued. Hence $\Phi(w,\mathbf{p}) = \Phi(-w,\mathbf{p}) = \Phi(w,-\mathbf{p}) = \Phi(-w,-\mathbf{p})$ which is related to the detailed balance of the limiting scattering operators described below.

The first main assumption is the Gaussian property of the random potential.

Assumption 1. *$V(z,\mathbf{x})$ is a Gaussian process with a spectral density $\Phi(\vec{\mathbf{k}}), \vec{\mathbf{k}} = (w,\mathbf{p}) \in \mathbb{R}^{d+1}$ which is smooth, uniformly bounded and decays at $|\vec{\mathbf{k}}| = \infty$ with sufficiently high power of $|\vec{\mathbf{k}}|^{-1}$.*

We note that the assumption of Gaussianity is not essential and is made here to simplify the presentation. Its main use is in deriving the estimates (40), (41) below.

Let $\mathcal{F}_z$ and $\mathcal{F}_z^+$ be the sigma-algebras generated by $\{V_s : \forall s \le z\}$ and $\{V_s : \forall s \ge z\}$, respectively and let $L^2(\mathcal{F}_z)$ and $L^2(\mathcal{F}_z^+)$ denote the square-integrable functions measurable w.r.t. to them respectively. The maximal correlation coefficient $\rho(t)$ is given by

$$\rho(t) = \sup_{\substack{h \in L^2(\mathcal{F}_z) \\ \mathbb{E}[h]=0, \mathbb{E}[h^2]=1}} \sup_{\substack{g \in L^2(\mathcal{F}_{z+t}^+) \\ \mathbb{E}[g]=0, \mathbb{E}[g^2]=1}} \mathbb{E}[hg]. \tag{15}$$

For Gaussian processes the correlation coefficient $\rho(t)$ equals the linear correlation coefficient given by

$$\sup_{g_1,g_2} \int R(t-\tau_1-\tau_2,\mathbf{k}) g_1(\tau_1,\mathbf{k}) g_2(\tau_2,\mathbf{k}) d\mathbf{k} d\tau_1 d\tau_2 \tag{16}$$

where $R(t,\mathbf{k}) = \int e^{it\xi}\Phi(\xi,\mathbf{k})d\xi$ and the supremum is taken over all $g_1, g_2 \in L^2(\mathbb{R}^{d+1})$ which are supported on $(-\infty, 0] \times \mathbb{R}^d$ and satisfy the constraint

$$\int R(t-t',\mathbf{k})g_1(t,\mathbf{k})g_1^*(t',\mathbf{k})dtdt'd\mathbf{k} = \int R(t-t',\mathbf{k})g_2(t,\mathbf{k})g_2^*(t',\mathbf{k})dtdt'd\mathbf{k} = 1.$$

There are various criteria for the decay rate of the linear correlation coefficients, see [12].

Next we make assumption on the mixing property of the random potential.

Assumption 2. *The maximal correlation coefficient $\rho(t)$ is integrable: $\int_0^\infty \rho(s)ds < \infty$.*

We have two main theorems depending on whether $\lim_{\varepsilon\to 0}\theta$ is positive or not.

Theorem 1. *Let $\theta > 0$ be fixed. Then under the two-frequency radiative transfer scaling (2)-(4), (10) the weak solutions, denoted by W_z^ε, of the Wigner-Moyal equation (8) converges in probability in $C([0,\infty), L^2_w(\mathbb{R}^d))$, the space of L^2-valued z-continuous processes, to that of the following deterministic equation*

$$\frac{\partial}{\partial z}W_z + \mathbf{p}\cdot\nabla W_z = \frac{2\pi k^2}{\theta^2}\int K(\mathbf{p},\mathbf{q})\left[e^{-i\beta\theta\mathbf{q}\cdot\mathbf{x}/(2\sqrt{k})}W_z(\mathbf{x},\mathbf{p}+\frac{\theta\mathbf{q}}{\sqrt{k}}) - W_z(\mathbf{x},\mathbf{p})\right]d\mathbf{q} \tag{17}$$

where the kernel K is given by

$$K(\mathbf{p},\mathbf{q}) = \Phi(0,\mathbf{q}), \quad \text{for} \quad \alpha \in (0,1),$$

and

$$K(\mathbf{p},\mathbf{q}) = \Phi\big((\mathbf{p}+\frac{\theta\mathbf{q}}{2\sqrt{k}})\cdot\mathbf{q},\mathbf{q}\big), \quad \text{for} \quad \alpha = 1.$$

Theorem 2. *Assume $\lim_{\varepsilon\to 0}\theta = 0$. Then under the two-frequency radiative transfer scaling (2)-(4), (10) the weak solutions of the Wigner-Moyal equation (8) converges in probability in the space $C([0,\infty), L^2_w(\mathbb{R}^d))$ to that of the following deterministic equation*

$$\frac{\partial}{\partial z}W_z + \mathbf{p}\cdot\nabla W_z = k\left(\nabla_\mathbf{p} - \frac{i}{2}\beta\mathbf{x}\right)\cdot\mathbf{D}\cdot\left(\nabla_\mathbf{p} - \frac{i}{2}\beta\mathbf{x}\right)W_z \tag{18}$$

where the (momentum) diffusion coefficient $\mathbf{D}$ is given by

$$\mathbf{D} = \pi\int\Phi(0,\mathbf{q})\mathbf{q}\otimes\mathbf{q}d\mathbf{q}, \quad \text{for} \quad \alpha\in(0,1), \tag{19}$$

$$\mathbf{D}(\mathbf{p}) = \pi\int\Phi(\mathbf{p}\cdot\mathbf{q},\mathbf{q})\mathbf{q}\otimes\mathbf{q}d\mathbf{q}, \quad \text{for} \quad \alpha = 1. \tag{20}$$

Remark 1. *In the* transverse *case of $\rho_x \ll \rho_z$ (or $\alpha > 1$), then with the choice of $\mu = \sqrt{\rho_x}$ (or ε^α) the limiting kernel and diffusion coefficient become, respectively*

$$\begin{aligned} K(\mathbf{p},\mathbf{q}) &= \delta\big((\mathbf{p}+\frac{\theta\mathbf{q}}{2\sqrt{k}})\cdot\mathbf{q}\big)\int\Phi(w,\mathbf{q})dw, \\ \mathbf{D}(\mathbf{p}) &= \pi|\mathbf{p}|^{-1}\int_{\mathbf{p}\cdot\mathbf{p}_\perp=0}\int\Phi(w,\mathbf{p}_\perp)dw\ \mathbf{p}_\perp\otimes\mathbf{p}_\perp d\mathbf{p}_\perp. \end{aligned} \tag{21}$$

The proof of such result requires additional assumptions which would complicate our presentation, so we will not pursue it here, cf. [8], [6].

Remark 2. *The form of (17) and (18) suggests the new quantity*

$$\mathfrak{W}_z(\mathbf{x},\mathbf{p}) = e^{-\frac{i\beta}{2}\mathbf{x}\cdot\mathbf{p}}W_z(\mathbf{x},\mathbf{p})$$

in terms of which the equations can be written respectively as

$$\frac{\partial}{\partial z}\mathfrak{W}_z+\mathbf{p}\cdot\nabla\mathfrak{W}_z+\frac{i\beta}{2}|\mathbf{p}|^2\mathfrak{W}_z = \frac{2\pi k^2}{\theta^2}\int K(\mathbf{p},\mathbf{q})\left[\mathfrak{W}_z(\mathbf{x},\mathbf{p}+\frac{\theta\mathbf{q}}{\sqrt{k}})-\mathfrak{W}_z(\mathbf{x},\mathbf{p})\right]d\mathbf{q}, \tag{22}$$

$$\frac{\partial}{\partial z}\mathfrak{W}_z+\mathbf{p}\cdot\nabla\mathfrak{W}_z+\frac{i\beta}{2}|\mathbf{p}|^2\mathfrak{W}_z = k\nabla_{\mathbf{p}}\cdot\mathbf{D}\cdot\nabla_{\mathbf{p}}\mathfrak{W}_z. \tag{23}$$

The advantage of this is that the solution then is amenable to probabilistic representation as the scattering terms on the right side of equations are non-positive definite. Let $(\mathbf{x}(z),\mathbf{p}(z))$ *be the stochastic process with* $\mathbf{x}(0)=\mathbf{x},\mathbf{p}(0)=\mathbf{p}$ *generated by the operator* $-\mathbf{p}\cdot\nabla_{\mathbf{x}}+\mathcal{A}$ *where* $\mathcal{A}$ *is the operator on the right hand side of either (22) or (23). Then by Dynkin's formula we have*

$$\mathfrak{W}_z(\mathbf{x},\mathbf{p})=\mathbb{E}_{\mathbf{x},\mathbf{p}}\left[e^{-\frac{i\beta}{2}\int_0^z|\mathbf{p}(t)|^2dt}e^{-\frac{i\beta}{2}\mathbf{x}(z)\cdot\mathbf{p}(z)}W_0(\mathbf{x}(z),\mathbf{p}(z))\right] \tag{24}$$

where W_0 *is the initial data for the two-frequency Wigner distribution and* $\mathbb{E}_{\mathbf{x},\mathbf{p}}$ *is the expectation with respect to the probability measure associated with* $(\mathbf{x}(z),\mathbf{p}(z))$*. The probabilistic solution (24) can be simulated by Monte Carlo method.*

When $k_1=k_2$ or $\beta=0$, eq. (17) and (18) reduce to the standard radiative transfer equations derived in [6], [8]. In view of the definition (5) the two-frequency Wigner distribution, captures the space-frequency correlation on the nondimensionalized scale of γ (in space as well as in frequency). The self-averaging property simply reflects the fact that on the wave field essentially decorrelates on the macroscopic scale $\gg\gamma$ and the rapidly fluctuating limit is then statistically stablized by coupling with a smooth deterministic test function.

A notable fact is that eq. (18) with (19) is the same governing equation, except for a constant damping term, for the *ensemble-averaged* two-frequency Wigner distribution for the z-white-noise potential in the geometrical optics regime, [7]. Fortunately eq. (18)-(19) is exactly solvable and the solution yields asymptotically precise information about the cross-frequency correlation, important for analyzing the information transfer and time reversal with broadband signals in the channel described by the stochastic Schrödinger equation with a z-white-noise potential [9] (see also [2], [3], [4]). Moreover, eq. (18)-(19) arises as the boundary layer equation for the two-frequency radiative transfer of *spherical wave* [10].

The proof of Theorem 2 is analogous to that of Theorem 1 and for the sake of space we will not repeat the argument. We refer the reader to [6] for the needed minor modification. More directly eq. (18) can be obtained from eq. (17) in the limit $\theta\to 0$ by Taylor expanding the terms involving θ on the right hand side of (17) up to second order in θ. The first-order-in-θ terms are also first-order-in-$\mathbf{q}$ and thus vanish due to the symmetry $K(\mathbf{p},\mathbf{q})=K(\mathbf{p},-\mathbf{q})$. The remaining terms, after dividing by θ^2 and passing to the limit, become the right hand side of (18).

For the proof of Theorem 1 below, we set $\theta=1$ for ease of notation.

3. Martingale formulation

For tightness as well as identification of the limit, the following infinitesimal operator $\mathcal{A}^\varepsilon$ will play an important role. Let $V_z^\varepsilon\equiv V(z/\varepsilon^2,\cdot)$. Let $\mathcal{F}_z^\varepsilon$ be the σ-algebras generated by $\{V_s^\varepsilon,\, s\leq z\}$ and $\mathbb{E}_z^\varepsilon$ the corresponding conditional expectation w.r.t. $\mathcal{F}_z^\varepsilon$. Let $\mathcal{M}^\varepsilon$ be the space of measurable function adapted to $\{\mathcal{F}_z^\varepsilon,\forall t\}$ such that $\sup_{z<z_0}\mathbb{E}|f(z)|<\infty$. We say $f(\cdot)\in\mathcal{D}(\mathcal{A}^\varepsilon)$, the domain of $\mathcal{A}^\varepsilon$, and $\mathcal{A}^\varepsilon f=g$ if $f,g\in\mathcal{M}^\varepsilon$ and for $f^\delta(z)\equiv\delta^{-1}[\mathbb{E}_z^\varepsilon f(z+\delta)-f(z)]$ we have

$$\begin{aligned}\sup_{z,\delta}\mathbb{E}|f^\delta(z)| &< \infty\\ \lim_{\delta\to 0}\mathbb{E}|f^\delta(z)-g(z)| &= 0,\quad\forall z.\end{aligned}$$

Consider a special class of admissible functions $f(z) = \phi(\langle W_z^\varepsilon, \Theta\rangle), f'(z) = \phi'(\langle W_z^\varepsilon, \Theta\rangle), \forall \phi \in C^\infty(\mathbb{R})$ we have the following expression from (12) and the chain rule

$$\mathcal{A}^\varepsilon f(z) \quad = \quad f'(z)\left[\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla\Theta\rangle + \frac{1}{\varepsilon}\langle W_z^\varepsilon, \mathcal{L}_z^*\Theta\rangle\right]. \tag{25}$$

In case of the test function Θ that is also a functional of the media we have

$$\mathcal{A}^\varepsilon f(z) \quad = \quad f'(z)\left[\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla\Theta\rangle + \frac{1}{\varepsilon}\langle W_z^\varepsilon, \mathcal{L}_z^*\Theta\rangle + \langle W_z^\varepsilon, \mathcal{A}^\varepsilon\Theta\rangle\right] \tag{26}$$

and when Θ depends explicitly on the fast spatial variable

$$\tilde{\mathbf{x}} = \mathbf{x}/\varepsilon^{2\alpha}$$

the gradient ∇ is a sum of two terms:

$$\nabla = \nabla_{\mathbf{x}} + \varepsilon^{-2\alpha}\nabla_{\tilde{\mathbf{x}}}$$

where $\nabla_{\mathbf{x}}$ is the gradient w.r.t. the slow variable $\mathbf{x}$ and $\nabla_{\tilde{\mathbf{x}}}$ the gradient w.r.t. the fast variable $\tilde{\mathbf{x}}$.

A main property of $\mathcal{A}^\varepsilon$ is that

$$f(z) - \int_0^z \mathcal{A}^\varepsilon f(s)ds \quad \text{is a } \mathcal{F}_z^\varepsilon\text{-martingale}, \quad \forall f \in \mathcal{D}(\mathcal{A}^\varepsilon). \tag{27}$$

Also,

$$\mathbb{E}_s^\varepsilon f(z) - f(s) = \int_s^z \mathbb{E}_s^\varepsilon \mathcal{A}^\varepsilon f(\tau)d\tau \quad \forall s < z \quad \text{a.s.} \tag{28}$$

(see [14]). We denote by $\mathcal{A}$ the infinitesimal operator corresponding to the unscaled process $V_z(\cdot) = V(z,\cdot)$.

4. Proof of tightness

In the sequel we will adopt the following notation

$$f(z) \equiv \phi(\langle W_z^\varepsilon, \Theta\rangle), \quad f'(z) \equiv \phi'(\langle W_z^\varepsilon, \Theta\rangle), \quad f''(z) \equiv \phi''(\langle W_z^\varepsilon, \Theta\rangle), \qquad \forall \phi \in C^\infty(\mathbb{R}).$$

Namely, the prime stands for the differentiation w.r.t. the original argument (not t) of f, f' etc.

A family of processes $\{W_z^\varepsilon, 0 < \varepsilon < 1\}$ in the Skorohod space $D([0,\infty); L_w^2(\mathbb{R}^{2d}))$ is tight if and only if the family of processes $\{\langle W_z^\varepsilon, \Theta\rangle, 0 < \varepsilon < 1\} \subset D([0,\infty); L_w^2(\mathbb{R}^{2d}))$ is tight for all $\Theta \in C_c^\infty$ [11]. We use the tightness criterion of [15] (Chap. 3, Theorem 4), namely, we will prove: Firstly,

$$\lim_{N\to\infty} \limsup_{\varepsilon\to 0} \mathbb{P}\{\sup_{z<z_0} |\langle W_z^\varepsilon, \Theta\rangle| \geq N\} = 0, \quad \forall z_0 < \infty. \tag{29}$$

Secondly, for each $\phi \in C^\infty(\mathbb{R})$ there is a sequence $f^\varepsilon(z) \in \mathcal{D}(\mathcal{A}^\varepsilon)$ such that for each $z_0 < \infty$ $\{\mathcal{A}^\varepsilon f^\varepsilon(z), 0 < \varepsilon < 1, 0 < z < z_0\}$ is uniformly integrable and

$$\lim_{\varepsilon\to 0} \mathbb{P}\{\sup_{z<z_0} |f^\varepsilon(z) - \phi(\langle W_z^\varepsilon, \Theta\rangle)| \geq \delta\} = 0, \quad \forall \delta > 0. \tag{30}$$

Then it follows that the laws of $\{\langle W_z^\varepsilon, \Theta\rangle, 0 < \varepsilon < 1\}$ are tight in the space of $D([0,\infty); \mathbb{R})$.

To prove the tightness in the space $C([0,\infty); L_w^2(\mathbb{R}^{2d}))$ let us recall that $W_z^\varepsilon \in C([0,\infty); L_w^2(\mathbb{R}^{2d}))$ and that the Skorohod metric and the uniform metric induce the same topology on $C([0,\infty); L_w^2(\mathbb{R}^{2d}))$.

Let us note first that condition (29) is satisfied because the L^2-norm is uniformly bounded. The rest of the argument for tightness will be concerned with establishing the second part of the criterion.

Consider now the expression

$$(31)\tilde{\mathcal{L}}^*_z\Theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) \equiv -i\varepsilon^{-2}\int_z^\infty\int\left[e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2\sqrt{k_1}})-e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k_2}})\right] \times e^{i(s-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}\mathbb{E}^\varepsilon_z\hat{V}^\varepsilon_s(d\mathbf{q})ds$$

or equivalently

$$(32)\qquad \mathcal{F}_2^{-1}\tilde{\mathcal{L}}^*_z\Theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y}) = \varepsilon^{-2}\int_z^\infty e^{-i\varepsilon^{-2\alpha}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}^\varepsilon_z\left[\delta_\varepsilon V^\varepsilon_s\right]\mathcal{F}_2^{-1}\Theta\right](\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})ds$$

where $\delta_\varepsilon V^\varepsilon_s$ is defined by (14). It is straightforward to check that (31) solves the corrector equation

$$(33)\qquad \left[\varepsilon^{-2\alpha}\mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}+\mathcal{A}^\varepsilon\right]\tilde{\mathcal{L}}^*_z\Theta=\varepsilon^{-2}\mathcal{L}^*_z\Theta$$

Recall that $\nabla_{\tilde{\mathbf{x}}}$ and $\nabla_{\mathbf{x}}$ are the gradients w.r.t. the fast variable $\tilde{\mathbf{x}}$ and the slow variable $\mathbf{x}$, respectively.

We have the fowllowing estimate.

Lemma 1.

$$\mathbb{E}\left[\tilde{\mathcal{L}}^*_z\Theta\right]^2(\mathbf{x},\mathbf{p}) \leq \left[\int_0^\infty\rho(s)ds\right]^2\int\left|e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k_1}})-e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2\sqrt{k_2}})\right|^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q}.$$

Proof. Consider the following trial functions in the definition of the maximal correlation coefficient

$$\begin{aligned} h &= h_s(\mathbf{x},\mathbf{p})\\ &= i\int\left[e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k_1}})-e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2\sqrt{k_2}})\right]e^{ik^{-1}(s-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}\mathbb{E}^\varepsilon_z\hat{V}^\varepsilon_s(d\mathbf{q})\\ g &= g_t(\mathbf{x},\mathbf{p})\\ &= i\int\left[e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k_1}})-e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2\sqrt{k_2}})\right]e^{ik^{-1}(t-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}\hat{V}^\varepsilon_t(d\mathbf{q}) \end{aligned}$$

It is easy to see that $h_s\in L^2(P,\Omega,\mathcal{F}_{\varepsilon^{-2}z})g_t\in\in L^2(P,\Omega,\mathcal{F}^+_{\varepsilon^{-2}t})$ and their second moments are uniformly bounded in $\mathbf{x},\mathbf{p},\varepsilon$ since

$$\begin{aligned} \mathbb{E}[h_s^2](\mathbf{x},\mathbf{p}) &\leq \mathbb{E}[g_s^2](\mathbf{x},\mathbf{p})\\ \mathbb{E}[g_s^2](\mathbf{x},\mathbf{p}) &= \int\left|e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k_1}})-e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2\sqrt{k_2}})\right|^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q} \end{aligned}$$

which is uniformly bounded for any integrable spectral density Φ.

From the definition (15) we have

$$|\mathbb{E}[h_s(\mathbf{x},\mathbf{p})h_t(\mathbf{y},\mathbf{q})]| = |\mathbb{E}\left[h_sg_t\right]| \leq \rho(\varepsilon^{-2}(t-z))\mathbb{E}^{1/2}\left[h_s^2(\mathbf{x},\mathbf{p})\right]\mathbb{E}^{1/2}\left[g_t^2(\mathbf{y},\mathbf{q})\right].$$

Hence by setting $s=t,\mathbf{x}=\mathbf{y},\mathbf{p}=\mathbf{q}$ first and the Cauchy-Schwartz inequality we have

$$\begin{aligned} \mathbb{E}\left[h_s^2(\mathbf{x},\mathbf{p})\right] &\leq \rho^2(\varepsilon^{-2}(s-z))\mathbb{E}[g_t^2(\mathbf{x},\mathbf{p})]\\ |\mathbb{E}\left[h_s(\mathbf{x},\mathbf{p})h_t(\mathbf{y},\mathbf{q})\right]| &\leq \rho(\varepsilon^{-2}(t-z))\rho(\varepsilon^{-2}(s-z))\mathbb{E}^{1/2}[g_t^2(\mathbf{x},\mathbf{p})]\mathbb{E}^{1/2}[g_t^2(\mathbf{y},\mathbf{q})],\quad \forall s,t\geq z,\forall\mathbf{x},\mathbf{y}. \end{aligned}$$

Hence

$$\varepsilon^{-4}\int_z^\infty\int_z^\infty\mathbb{E}[h_s(\mathbf{x},\mathbf{p})g_t(\mathbf{x},\mathbf{p})]dsdt \leq \mathbb{E}[g_t^2](\mathbf{x},\mathbf{p})\left[\int_0^\infty\rho(s)ds\right]^2$$

which together with (34) yields the lemma. □

Corollary 1.

$$\mathbb{E}\left[\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^*\Theta\right]^2(\mathbf{x},\mathbf{p}) \leq \left[\int_0^\infty \rho(s)ds\right]^2 \int \Big| e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}} k_1 \mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k_1}}) - e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}} k_2 \mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2\sqrt{k_2}})\Big|^2 \Phi(\xi,\mathbf{q})d\xi d\mathbf{q}. \tag{34}$$

Inequality (34) can be obtained from the expression

$$\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^*\Theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) \equiv i\varepsilon^{-2}\int_z^\infty \int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}\Big[e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}} k_2\mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k_2}}) - e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}} k_1\mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2\sqrt{k_1}})\Big] e^{i(s-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}\mathbb{E}_z^\varepsilon\hat{V}_s^\varepsilon(d\mathbf{q})ds$$

as in Lemma 1.

We will need to estimate the iteration of $\mathcal{L}_z$ and $\tilde{\mathcal{L}}_z^*$:

$$\begin{aligned}
&\mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\Theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p})\\
&= -\frac{1}{\varepsilon^2}\int_z^\infty ds\int \hat{V}_z^\varepsilon(d\mathbf{q})\mathbb{E}_z^\varepsilon[\hat{V}_s^\varepsilon(d\mathbf{q}')]e^{i(s-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}\Big\{\Big[e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}e^{i\mathbf{q}'\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}\\
&\quad\times k_1^2\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}'}{2\sqrt{k_1}}+\frac{\mathbf{q}}{2\sqrt{k_1}}) - e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}e^{i\mathbf{q}'\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_1k_2\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}'}{2\sqrt{k_2}}+\frac{\mathbf{q}}{2\sqrt{k_1}})\Big]e^{i(s-z)\mathbf{q}'\cdot\mathbf{q}/(2\varepsilon^{2\alpha})}\\
&\quad-\Big[e^{i\mathbf{q}\tilde{\mathbf{x}}/\sqrt{k_2}}e^{i\mathbf{q}'\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1k_2\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}'}{2\sqrt{k_1}}-\frac{\mathbf{q}}{2\sqrt{k_2}}) - e^{i\mathbf{q}\tilde{\mathbf{x}}/\sqrt{k_2}}e^{i\mathbf{q}'\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2^2\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}'}{2\sqrt{k_2}}-\frac{\mathbf{q}}{2\sqrt{k_2}})\Big]\\
&\quad\times e^{-i(s-z)\mathbf{q}'\cdot\mathbf{q}/(2\varepsilon^{2\alpha})}\Big\}\\
&\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p})\\
&= -\frac{1}{\varepsilon^4}\int_z^\infty\int_z^\infty dsdt\int \mathbb{E}_z^\varepsilon\hat{V}_s^\varepsilon(d\mathbf{q})\mathbb{E}_z^\varepsilon[\hat{V}_t^\varepsilon(d\mathbf{q}')]e^{i(s-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}e^{i(t-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}\Big\{\Big[e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}e^{i\mathbf{q}'\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}\\
&\quad\times k_1^2\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}'}{2\sqrt{k_1}}+\frac{\mathbf{q}}{2\sqrt{k_1}}) - e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}e^{i\mathbf{q}'\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_1k_2\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}'}{2\sqrt{k_2}}+\frac{\mathbf{q}}{2\sqrt{k_1}})\Big]e^{i(s-z)\mathbf{q}'\cdot\mathbf{q}/(2\varepsilon^{2\alpha})}\\
&\quad-\Big[e^{i\mathbf{q}\tilde{\mathbf{x}}/\sqrt{k_2}}e^{i\mathbf{q}'\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1k_2\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}'}{2\sqrt{k_1}}-\frac{\mathbf{q}}{2\sqrt{k_2}}) - e^{i\mathbf{q}\tilde{\mathbf{x}}/\sqrt{k_2}}e^{i\mathbf{q}'\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2^2\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}'}{2\sqrt{k_2}}-\frac{\mathbf{q}}{2\sqrt{k_2}})\Big]\\
&\quad\times e^{-i(s-z)\mathbf{q}'\cdot\mathbf{q}/(2\varepsilon^{2\alpha})}\Big\}
\end{aligned}$$

which can be more easily estimated by using (32) as follows. First we have the expressions after the inverse Fourier transform

$$\mathcal{F}_2^{-1}\Big\{\mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\Theta\Big\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y}) = \varepsilon^{-2}\int_z^\infty \delta_\varepsilon V_z^\varepsilon e^{-i\varepsilon^{-2\alpha}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\Theta\right](\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})ds \tag{35}$$

$$\mathcal{F}_2^{-1}\Big\{\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\Big\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y}) = -\varepsilon^{-4}\int_z^\infty e^{-i\varepsilon^{-2\alpha}(t-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\Big\{\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon]e^{-i\varepsilon^{-2\alpha}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}} \left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\Theta\right]\Big\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})dsdt. \tag{36}$$

Lemma 2.

$$
\begin{aligned}
\mathbb{E}\|\mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\Theta\|_2^2 &\le C\left(\int_0^\infty \rho(s)ds\right)^2 \mathbb{E}[V_z]^2 \int \Big| e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}} k_1 \Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k_1}}) \\
&\quad - e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}} k_2 \Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2\sqrt{k_2}})\Big|^2 \Phi(\xi,\mathbf{q}) d\xi d\mathbf{x} d\mathbf{q} d\mathbf{p} \\
\mathbb{E}\|\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\|_2^2 &\le C\left(\int_0^\infty \rho(s)ds\right)^4 \mathbb{E}[V_z]^2 \int \Big| e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}} k_1 \Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k_1}}) \\
&\quad - e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}} k_2 \Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2\sqrt{k_2}})\Big|^2 \Phi(\xi,\mathbf{q}) d\xi d\mathbf{x} d\mathbf{q} d\mathbf{p}
\end{aligned}
$$

for some constant C independent of ε.

Proof. Let us consider $\tilde{\mathcal{L}}_z^*\mathcal{L}_z^*\Theta$ here. The calculation for $\mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\Theta$ is similar.

By the Parseval theorem and the unitarity of $\exp(i\tau\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}), \tau\in\mathbb{R}$,

$$
\begin{aligned}
&\mathbb{E}\|\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\|_2^2 = \mathbb{E}\|\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\|_2^2 \\
&\le C_0\varepsilon^{-8}\int\int_z^\infty \Big|\mathbb{E}\Big\{\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon] e^{-i\varepsilon^{-2\alpha}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\Theta\right]\Big\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})\Big| dsdt \\
&\quad \int_z^\infty \Big|\mathbb{E}\Big\{\mathbb{E}_z[\delta_\varepsilon V_{t'}^\varepsilon] e^{i\varepsilon^{-2\alpha}(s'-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_{s'}^\varepsilon]\mathcal{F}_2^{-1}\Theta\right]\Big\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})\Big| ds'dt'd\mathbf{x}d\mathbf{y} \\
&\quad + C_0\varepsilon^{-8}\int\int_z^\infty \left|\mathbb{E}\left[\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon]\mathbb{E}_z[\delta_\varepsilon V_{t'}^\varepsilon]\right]\right| \Big|\mathbb{E}\Big\{e^{-i\varepsilon^{-2\alpha}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\Theta\right](\mathbf{x},\mathbf{y}) \\
&\qquad \times e^{i\varepsilon^{-2\alpha}(s'-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_{s'}^\varepsilon]\mathcal{F}_2^{-1}\Theta\right](\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})\Big\}\Big| dsdtds'dt'd\mathbf{x}d\mathbf{y}.
\end{aligned}
$$

The last inequality follows from the Gaussian property. Note that in the $\mathbf{x}$ integrals above the fast variable $\tilde{\mathbf{x}}$ is integrated and is not treated as independent of $\mathbf{x}$.

Let

$$
g(t) = \delta_\varepsilon V_t^\varepsilon
$$

and

$$
h(s) = e^{-i\varepsilon^{-2\alpha}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\delta_\varepsilon V_s^\varepsilon \mathcal{F}_2^{-1}\Theta\right].
$$

The same argument as that for Lemma 1 yields

$$
\begin{aligned}
|\mathbb{E}[\mathbb{E}_z[g(t)]\mathbb{E}_z[h(s)]]| &\le \mathbb{E}^{1/2}[\mathbb{E}_z[g(t)]^2]\mathbb{E}^{1/2}[\mathbb{E}_z[h(s)]^2] \\
&\le \rho(\varepsilon^{-2}(t-z))\rho(\varepsilon^{-2}(s-z))\mathbb{E}^{1/2}[g^2(t)]\mathbb{E}^{1/2}[h^2(s)], \quad t,s\ge z; \\
\left|\mathbb{E}[\mathbb{E}_z[g(t)]\mathbb{E}_z[g(t')]]\right| &\le \mathbb{E}^{1/2}[\mathbb{E}_z[g(t)]^2]\mathbb{E}^{1/2}[\mathbb{E}_z[g(t')]^2] \\
&\le \rho(\varepsilon^{-2}(t-z))\rho(\varepsilon^{-2}(t'-z))\mathbb{E}^{1/2}[g^2(t)]\mathbb{E}^{1/2}[g^2(t')], \quad t,t'\ge z; \\
\left|\mathbb{E}[\mathbb{E}_z[h(s)]\mathbb{E}_z[h(s')]]\right| &\le \mathbb{E}^{1/2}[\mathbb{E}_z[h(s)]^2]\mathbb{E}^{1/2}[\mathbb{E}_z[h(s')]^2] \\
&\le \rho(\varepsilon^{-2}(s-z))\rho(\varepsilon^{-2}(s'-z))\mathbb{E}^{1/2}[h^2(s)]\mathbb{E}^{1/2}[h^2(s')], \quad s,s'\ge z.
\end{aligned}
$$

Combining the above estimates we get

$$
\begin{aligned}
\mathbb{E}\|\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\|_2^2 &\le C_1\left(\int_0^\infty \rho(s)ds\right)^4 \int \mathbb{E}[\delta_\varepsilon V_z^\varepsilon]^2 \mathbb{E}\left[e^{-i\varepsilon^{-2\alpha}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\delta_\varepsilon V_s^\varepsilon\mathcal{F}_2^{-1}\Theta\right]\right]^2 d\mathbf{x}d\mathbf{y} \\
&\le C_2\left(\int_0^\infty \rho(s)ds\right)^4 \mathbb{E}[V_z^\varepsilon]^2 \int \Big| e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}} k_1 \nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k_1}}) \\
&\quad - e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}} k_2 \nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2\sqrt{k_2}})\Big|^2 \Phi(\xi,\mathbf{q}) d\xi d\mathbf{x} d\mathbf{q} d\mathbf{p}
\end{aligned}
$$

$\square$

Now let us consider the second moment of $\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta$ and $\mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta$:

$$
\begin{aligned}
&\mathcal{F}_2^{-1}\left\{\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\right\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})\\
&= i\varepsilon^{-2}\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{x}}\int_z^\infty e^{-i\varepsilon^{-2\alpha}(t-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left\{\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon]e^{-i\varepsilon^{-2\alpha}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\Theta\right]\right\}(\mathbf{x},\mathbf{y})dsdt\\
&= i\varepsilon^{-2}\int_z^\infty e^{-i\varepsilon^{-2\alpha}(t-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left\{\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon]e^{-i\varepsilon^{-2\alpha}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\nabla_{\mathbf{y}}\delta_\varepsilon V_s^\varepsilon]\cdot\mathcal{F}_2^{-1}\nabla_{\mathbf{x}}\Theta\right]\right\}(\mathbf{x},\mathbf{y})dsdt\\
&\quad +i\varepsilon^{-2}\int_z^\infty e^{-i\varepsilon^{-2\alpha}(t-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left\{\mathbb{E}_z[\nabla_{\mathbf{y}}\delta_\varepsilon V_t^\varepsilon]\cdot e^{-i\varepsilon^{-2\alpha}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\nabla_{\mathbf{x}}\Theta\right]\right\}(\mathbf{x},\mathbf{y})dsdt
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{F}_2^{-1}\left\{\mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\right\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})\\
&= i\varepsilon^{-4}\delta_\varepsilon V_z^\varepsilon(\tilde{\mathbf{x}},\mathbf{y})\int_z^\infty e^{-i\varepsilon^{-2\alpha}(t-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left\{\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon]e^{-i\varepsilon^{-2\alpha}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\Theta\right]\right\}(\mathbf{x},\mathbf{y})dsdt.
\end{aligned}
$$

The same calculation as in Lemma 2 yields the following estimates:

Corollary 2.

$$
\begin{aligned}
\mathbb{E}\|\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\|_2^2 \leq\; & C\left(\int_0^\infty\rho(s)ds\right)^4\Big\{\mathbb{E}[\nabla_{\mathbf{y}}V_z^\varepsilon]^2\int\Big|e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k_1}})\\
&-e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2\sqrt{k_2}})\Big|^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}\\
&+\mathbb{E}[V_z^\varepsilon]^2\int\Big|e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1\mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k_1}})\\
&-e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2\mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2\sqrt{k_2}})\Big|^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}\Big\};
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\|\mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\|_2^2 \leq\; & C\left(\int_0^\infty\rho(s)ds\right)^4\mathbb{E}[V_z^\varepsilon]^4\int\Big|e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k_1}})\\
&-e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2\sqrt{k_2}})\Big|^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}
\end{aligned}
$$

for some constant C independent of ε.

Let

$$f_1(z)=\varepsilon f'(z)\left\langle W_z^\varepsilon,\tilde{\mathcal{L}}_z^*\Theta\right\rangle \tag{37}$$

be the 1-st perturbation of $f(z)$.

Proposition 1.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|f_1(z)|=0,\quad \lim_{\varepsilon\to 0}\sup_{z<z_0}|f_1(z)|=0\quad \textit{in probability}$$

.

Proof. We have

$$\mathbb{E}[|f_1(z)|]\leq\varepsilon\|f'\|_\infty\|W_0\|_2\mathbb{E}\|\tilde{\mathcal{L}}_z^*\Theta\|_2 \tag{38}$$

and

$$\sup_{z<z_0}|f_1^\varepsilon(z)|\leq\varepsilon\|f'\|_\infty\|W_0\|_2\sup_{z<z_0}\|\tilde{\mathcal{L}}_z^*\Theta\|_2. \tag{39}$$

Since $\tilde{\mathcal{L}}_z^*\Theta$ is a Gaussian process and $\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta$ is a χ^2-process, by an application of Borell's inequality [1] we have

$$\begin{aligned}
&(40) \qquad \mathbb{E}[\sup_{z<z_0} \|\tilde{\mathcal{L}}_z^*\Theta\|_2^2] \quad \leq \quad C\log(\frac{1}{\varepsilon})\mathbb{E}\|\tilde{\mathcal{L}}_z^*\Theta\|_2^2;\\
&(41) \qquad \mathbb{E}[\sup_{z<z_0} \|\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\|_2^2] \quad \leq \quad C\log^2(\frac{1}{\varepsilon})\mathbb{E}\|\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\|_2^2,
\end{aligned}$$

i.e. the supremum over $z < z_0$ inside the expectation can be over-estimated by a $\log(1/\varepsilon)$ factor for excursion on the scale of any power of $1/\varepsilon$. Hence the right side of (38) is $O(\varepsilon)$ while the right side of (39) is $o(1)$ in probability by Chebyshev's inequality. □

Set $f^\varepsilon(z) = f(z) + f_1(z)$. Then (30) follows immediately from Proposition 1.

Let us now prove the uniform integrability of $\mathcal{A}^\varepsilon f^\varepsilon$. A straightforward calculation yields

$$\begin{aligned}
\mathcal{A}^\varepsilon f_1 \quad = \quad & \varepsilon f'(z)\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^*\Theta\right\rangle + \varepsilon f''(z)\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta\right\rangle\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}^*\Theta\right\rangle\\
&+f'(z)\left\langle W_z^\varepsilon, \mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\Theta\right\rangle + f''(z)\left\langle W_z^\varepsilon, \mathcal{L}_z^*\Theta\right\rangle\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^*\Theta\right\rangle - \frac{1}{\varepsilon}f'(z)\left\langle W_z^\varepsilon, \mathcal{L}_z^*\Theta\right\rangle
\end{aligned}$$

and, hence

$$\begin{aligned}
(42) \quad \mathcal{A}^\varepsilon f^\varepsilon(z) \quad = \quad & f'(z)\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta\right\rangle + f'(z)\left\langle W_z^\varepsilon, \mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\Theta\right\rangle + f''(z)\left\langle W_z^\varepsilon, \mathcal{L}_z^*\Theta\right\rangle\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^*\Theta\right\rangle\\
&+\varepsilon\left[f'(z)\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^*\Theta\right\rangle + f''(z)\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta\right\rangle\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^*\Theta\right\rangle\right]\\
= \quad & A_0(z) + A_1(z) + A_2(z) + R_1(z)
\end{aligned}$$

where $A_1(z)$ and $A_2(z)$ are the $O(1)$ statistical coupling terms.

Proposition 2.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R_1(z)|^2 = 0$$

.

Proof. First we note that

$$|R_1| \quad \leq \quad \varepsilon\left[\|f''\|_\infty\|W_0\|_2^2\|\mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta\|_2\|\tilde{\mathcal{L}}_z^*\Theta\|_2 + \|f'\|_\infty\|W_z^\varepsilon\|_2\|\mathbf{p}\cdot\nabla_{\mathbf{x}}(\tilde{\mathcal{L}}_z^*\Theta)\|_2\right].$$

Clearly we have

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R_1(z)|^2 = 0.$$

by Lemma 1 and Corollary 1. □

For the tightness criterion stated in the beginnings of the section, it remains to show

Proposition 3. $\{\mathcal{A}^\varepsilon f^\varepsilon\}$ *are uniformly integrable.*

Proof. Let us show first that $\{A_i\}, i = 0, 1, 2, 3$ are uniformly integrable.

For this we have the following estimates:

$$\begin{aligned}
|A_0(z)| \quad &\leq \quad \|f'\|_\infty\|W_0\|_2\|\mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta\|_2\\
|A_1(z)| \quad &\leq \quad \|f'\|_\infty\|W_0\|_2\|\mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\Theta\|_2\\
|A_2(z)| \quad &\leq \quad \|f''\|_\infty\|W_0\|_2^2\|\mathcal{L}_z^*\Theta\|_2\|\tilde{\mathcal{L}}_z^*\Theta\|_2.
\end{aligned}$$

The second moments of the right hand side of the above expressions are uniformly bounded as $\varepsilon \to 0$ by Lemmas 1 and 2 and hence $A_0(z), A_1(z), A_2(z)$ are uniformly integrable. By Proposition 2, R_1 is uniformly integrable. Therefore $\mathcal{A}^\varepsilon f^\varepsilon$ is uniformly integrable by (42). □

5. Identification of the limit

The tightness just established permits passing to the weak limit. Our strategy for identifying the limit is to show directly that in passing to the weak limit the limiting process solves the martingale problem with *null* quadratic variation. This would imply the limiting equation is deterministic. The uniqueness of solution to the limiting deterministic equation for given data then identifies the limit.

For this purpose, we introduce the next perturbations f_2, f_3. Let

$$A_2^{(1)}(\psi) \equiv \int \psi(\mathbf{x},\mathbf{p})\mathcal{Q}_1(\Theta\otimes\Theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\psi(\mathbf{y},\mathbf{q})\, d\mathbf{x}d\mathbf{p}\, d\mathbf{y}d\mathbf{q} \tag{43}$$

$$A_1^{(1)}(\psi) \equiv \int \mathcal{Q}_1'\Theta(\mathbf{x},\mathbf{p})\psi(\mathbf{x},\mathbf{p})\; d\mathbf{x}d\mathbf{p}, \quad \forall \psi\in L^2(\mathbb{R}^{2d}) \tag{44}$$

where

$$\mathcal{Q}_1(\Theta\otimes\Theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q}) = \mathbb{E}\left[\mathcal{L}_z^*\Theta(\mathbf{x},\mathbf{p})\tilde{\mathcal{L}}_z^*\Theta(\mathbf{y},\mathbf{q})\right]$$

and

$$\mathcal{Q}_1'\Theta(\mathbf{x},\mathbf{p}) = \mathbb{E}\left[\mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\Theta(\mathbf{x},\mathbf{p})\right].$$

Clearly,

$$A_2^{(1)}(\psi) = \mathbb{E}\left[\langle\psi,\mathcal{L}_z^*\Theta\rangle\left\langle\psi,\tilde{\mathcal{L}}_z^*\Theta\right\rangle\right].$$

Let

$$\mathcal{Q}_2(\Theta\otimes\Theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q}) \equiv \mathbb{E}\left[\tilde{\mathcal{L}}_z^*\Theta(\mathbf{x},\mathbf{p})\tilde{\mathcal{L}}_z^*\Theta(\mathbf{y},\mathbf{q})\right]$$

and

$$\mathcal{Q}_2'\Theta(\mathbf{x},\mathbf{p}) = \mathbb{E}\left[\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta(\mathbf{x},\mathbf{p})\right].$$

Let

$$\begin{aligned} A_2^{(2)}(\psi) &\equiv \int \psi(\mathbf{x},\mathbf{p})\mathcal{Q}_2(\Theta\otimes\Theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\psi(\mathbf{y},\mathbf{q})\; d\mathbf{x}d\mathbf{p}\, d\mathbf{y}d\mathbf{q} \\ A_1^{(2)}(\psi) &\equiv \int \mathcal{Q}_2'\Theta(\mathbf{x},\mathbf{p})\psi(\mathbf{x},\mathbf{p})\; d\mathbf{x}\, d\mathbf{p} \end{aligned}$$

Define

$$f_2(z) = \frac{\varepsilon^2}{2}f''(z)\left[\left\langle W_z^\varepsilon,\tilde{\mathcal{L}}_z^*\Theta\right\rangle^2 - A_2^{(2)}(W_z^\varepsilon)\right] \tag{45}$$

$$f_3(z) = \frac{\varepsilon^2}{2}f'(z)\left[\left\langle W_z^\varepsilon,\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\right\rangle - A_1^{(2)}(W_z^\varepsilon)\right]. \tag{46}$$

Proposition 4.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|f_2(z)| = 0, \quad \lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|f_3(z)| = 0.$$

Proof. We have the bounds

$$\begin{aligned} \sup_{z<z_0}\mathbb{E}|f_2(z)| &\le \sup_{z<z_0}\varepsilon^2\|f''\|_\infty\left[\|W_0\|_2^2\mathbb{E}\|\tilde{\mathcal{L}}_z^*\Theta\|_2^2 + \mathbb{E}[A_2^{(2)}(W_z^\varepsilon)]\right] \\ \sup_{z<z_0}\mathbb{E}|f_3^\varepsilon(z)| &\le \sup_{z<z_0}\varepsilon^2\|f'\|_\infty\left[\|W_0\|_2\mathbb{E}\|\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\|_2 + \mathbb{E}[A_1^{(2)}(W_z^\varepsilon)]\right]. \end{aligned}$$

A straightforward calculation shows that $\mathbb{E}[A_2^{(2)}(W_z^\varepsilon)]$ and $\mathbb{E}[A_1^{(2)}(W_z^\varepsilon)]$ stay uniformly bounded w.r.t. ε. The right side of the above expressions then tends to zero as $\varepsilon\to 0$ by Lemma 1 and 2. □

We have

$$\mathcal{A}^\varepsilon f_2(z) = f''(z)\left[-\langle W_z^\varepsilon, \mathcal{L}_z^*\Theta\rangle \left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^*\Theta\right\rangle + A_2^{(1)}(W_z^\varepsilon)\right] + R_2(z)$$

$$\mathcal{A}^\varepsilon f_3(z) = f'(z)\left[-\left\langle W_z^\varepsilon, \mathcal{L}_z^*(\tilde{\mathcal{L}}_z^*\Theta)\right\rangle + A_1^{(1)}(W_z^\varepsilon)\right] + R_3(z)$$

with

$$\begin{aligned} R_2(z) &= \varepsilon^2 \frac{f'''(z)}{2}\left[\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta\rangle + \frac{1}{\varepsilon}\langle W_z^\varepsilon, \mathcal{L}_z^*\Theta\rangle\right]\left[\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^*\Theta\right\rangle^2 - A_2^{(2)}(W_z^\varepsilon)\right] \\ &\quad +\varepsilon^2 f''(z)\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^*\Theta\right\rangle\left[\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(\tilde{\mathcal{L}}_z^*\Theta)\right\rangle + \frac{1}{\varepsilon}\left\langle W_z^\varepsilon, \mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\Theta\right\rangle\right] \\ &\quad -\varepsilon^2 f''(z)\left[\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(G_\Theta^{(2)}W_z^\varepsilon)\right\rangle + \frac{1}{\varepsilon}\left\langle W_z^\varepsilon, \mathcal{L}_z^*G_\Theta^{(2)}W_z^\varepsilon\right\rangle\right] \end{aligned} \tag{47}$$

where $G_\Theta^{(2)}$ denotes the operator

$$G_\Theta^{(2)}\psi \equiv \int \mathcal{Q}_2(\Theta\otimes\Theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\psi(\mathbf{y},\mathbf{q})\,d\mathbf{y}d\mathbf{q}.$$

Similarly

$$\begin{aligned} R_3(z) &= \varepsilon^2 f'(z)\left[\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta)\right\rangle + \frac{k}{\varepsilon}\left\langle W_z^\varepsilon, \mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\right\rangle\right] \\ &\quad +\frac{\varepsilon^2}{2}f''(z)\left[\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta\rangle + \frac{1}{\varepsilon}\langle W_z^\varepsilon, \mathcal{L}_z^*\Theta\rangle\right]\left[\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\right\rangle - A_1^{(2)}(W_z^\varepsilon)\right] \\ &\quad -\varepsilon^2 f'(z)\left[\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(\mathcal{Q}_2'\Theta)\rangle + \frac{1}{\varepsilon}\langle W_z^\varepsilon, \mathcal{L}_z^*\mathcal{Q}_2'\Theta\rangle\right]. \end{aligned} \tag{48}$$

Proposition 5.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R_2(z)| = 0, \quad \lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R_3(z)| = 0.$$

Proof. Part of the argument is analogous to that given for Proposition 4. The additional estimates that we need to consider are the following.

In R_2: First we have

$$\begin{aligned} &\sup_{z<z_0}\varepsilon^2\mathbb{E}\left|\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(G_\Theta^{(2)}W_z^\varepsilon)\right\rangle\right| \\ &= \varepsilon^2\int \mathbb{E}\left[W_z^\varepsilon(\mathbf{x},\mathbf{p})W_z^\varepsilon(\mathbf{y},\mathbf{q})\right]\mathbb{E}\left[\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^*\Theta(\mathbf{x},\mathbf{p})\tilde{\mathcal{L}}_z^*\Theta(\mathbf{y},\mathbf{q})\right]d\mathbf{x}d\mathbf{y}d\mathbf{p}d\mathbf{q} \\ &\leq \varepsilon^2\int \mathbb{E}\left[W_z^\varepsilon(\mathbf{x},\mathbf{p})W_z^\varepsilon(\mathbf{y},\mathbf{q})\right]\mathbb{E}^{1/2}[\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^*\Theta]^2(\mathbf{x},\mathbf{p})\mathbb{E}^{1/2}[\tilde{\mathcal{L}}_z^*\Theta]^2(\mathbf{y},\mathbf{q})d\mathbf{x}d\mathbf{y}d\mathbf{p}d\mathbf{q} \end{aligned}$$

which is $O(\varepsilon^2)$ by using Lemma 1, Corollary 1 and the fact $\mathbb{E}\left[W_z^\varepsilon(\mathbf{x},\mathbf{p})W_z^\varepsilon(\mathbf{y},\mathbf{q})\right] \in L^2(\mathbb{R}^{4d})$ in conjunction with the same argument as in proof of Lemma 1; Secondly, we have

$$\begin{aligned} \sup_{z<z_0}\varepsilon\mathbb{E}\left|\left\langle W_z^\varepsilon, \mathcal{L}_z^*G_\Theta^{(2)}W_z^\varepsilon\right\rangle\right| &= \sup_{z<z_0}\varepsilon\|W_0\|_2\mathbb{E}\|\mathcal{L}_z^*\mathbb{E}\left[\tilde{\mathcal{L}}_z^*\Theta\otimes\tilde{\mathcal{L}}_z^*\Theta\right]W_z^\varepsilon\|_2 \\ &= \sup_{z<z_0}\varepsilon\|W_0\|_2\mathbb{E}\|\mathcal{F}_2^{-1}\mathcal{L}_z^*\mathbb{E}\left[\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^*\Theta\otimes\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^*\Theta\right]\mathcal{F}_2^{-1}W_z^\varepsilon\|_2. \end{aligned}$$

Define

$$h_s = e^{-i\varepsilon^{-2\alpha}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}[\delta_\varepsilon V_z^\varepsilon\mathcal{F}_2^{-1}\Theta].$$

We then have

$$\mathbb{E}\|\mathcal{F}_2^{-1}\mathcal{L}_z^*\mathbb{E}\left[\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^*\Theta\otimes\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^*\Theta\right]\mathcal{F}_2^{-1}W_z^\varepsilon\|_2$$

$$\begin{aligned}
&= \mathbb{E}\left\{\int\left|\varepsilon^{-4}\int\int_z^\infty \delta_\varepsilon V_z^\varepsilon(\mathbf{x},\mathbf{y})\mathbb{E}\left[\mathbb{E}_z[h_s(\mathbf{x},\mathbf{y})]\mathbb{E}_z[h_t(d\mathbf{x}',d\mathbf{y}')]\right]\mathcal{F}_2^{-1}W_z^\varepsilon(\mathbf{x}',\mathbf{y}')d\mathbf{x}'d\mathbf{y}'dsdt\right|^2 d\mathbf{x}d\mathbf{y}\right\}^{1/2}\\
&\le \mathbb{E}^{1/2}\left\{\int\left|\varepsilon^{-4}\int_z^\infty |\delta_\varepsilon V_z^\varepsilon(\mathbf{x},\mathbf{y})|\rho(\varepsilon^{-2}(s-z))\rho(\varepsilon^{-2}(t-z))\mathbb{E}^{1/2}[h_s(\mathbf{x},\mathbf{y})]^2\right.\right.\\
&\qquad\left.\left.\left(\int\mathbb{E}[h_t(d\mathbf{x}',d\mathbf{y}')]^2d\mathbf{x}'d\mathbf{y}'\right)\left(\int|W_z^\varepsilon(\mathbf{x}',\mathbf{p}')|^2d\mathbf{x}'d\mathbf{p}'\right)dsdt\right|^2 d\mathbf{x}d\mathbf{y}\right\}.
\end{aligned}$$

Recall that $\|W_z^\varepsilon\|_2\le\|W_0\|_2$ and

$$\int\mathbb{E}[h_t(d\mathbf{x}',d\mathbf{y}')]^2d\mathbf{x}'d\mathbf{y}'=\int[\Theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\Theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q}d\mathbf{x}d\mathbf{p}<\infty$$

so that

$$\begin{aligned}
&\mathbb{E}\|\mathcal{F}_2^{-1}\mathcal{L}_z^*\mathbb{E}\left[\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^*\Theta\otimes\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^*\Theta\right]\mathcal{F}_2^{-1}W_z^\varepsilon\|_2\\
&\le \|W_0\|_2\mathbb{E}^{1/2}\|h_s\|_2^2\left(\sup_{\mathbf{x},\mathbf{y}}\mathbb{E}[\delta_\varepsilon V_z^\varepsilon]^2\right)\varepsilon^{-8}\int_z^\infty\rho(\varepsilon^{-2}(s-z))\rho(\varepsilon^{-2}(t-z))\\
&\qquad\times\rho(\varepsilon^{-2}(s'-z))\rho(\varepsilon^{-2}(t'-z))\mathbb{E}^{1/2}\|h_s\|_2^2\mathbb{E}^{1/2}\|h_{s'}\|_2^2dsdtds'dt'\\
&\le \|W_0\|_2\mathbb{E}^{3/2}\|h_s\|_2^2\left(\sup_{\mathbf{x},\mathbf{y}}\mathbb{E}[\delta_\varepsilon V_z^\varepsilon]^2\right)\left|\int_0^\infty\rho(s)ds\right|^2<\infty.
\end{aligned}$$

Recall from (34) that

$$\mathbb{E}\|h_s\|_2^2 = \int[\Theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\Theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q}d\mathbf{x}d\mathbf{p}<\infty.$$

Hence

$$\sup_{z<z_0}\varepsilon\mathbb{E}\left|\left\langle W_z^\varepsilon,\mathcal{L}_z^*G_\Theta^{(2)}W_z^\varepsilon\right\rangle\right|=O(\varepsilon).$$

In R_3^ε:

$$\sup_{z<z_0}\varepsilon\mathbb{E}\left|\left\langle W_z^\varepsilon,\mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\right\rangle\right| \le \varepsilon\|W_0\|_2\sup_{z<z_0}\mathbb{E}\|\mathcal{L}_z^*\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\|_2$$

which is $O(\varepsilon)$ by Corollary 2.

The other two terms in R_3 have the respective expressions

$$\begin{aligned}
\varepsilon^2\mathbb{E}\left|\left\langle W_z^\varepsilon,\mathbf{p}\cdot\nabla_\mathbf{x}(\mathcal{Q}_2'\Theta)\right\rangle\right| &\le \varepsilon^2\|W_0\|_2\mathbb{E}^{1/2}\|\mathbf{p}\cdot\nabla_\mathbf{x}\mathbb{E}[\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta]\|_2^2\\
&\le \varepsilon^2\|W_0\|_2\|\mathbb{E}[\mathbf{p}\cdot\nabla_\mathbf{x}\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta]\|_2
\end{aligned}$$

which is $O(\varepsilon^2)$ by Corollary 2 and

$$\begin{aligned}
\varepsilon\mathbb{E}\left|\left\langle W_z^\varepsilon,\mathcal{L}_z^*\mathcal{Q}_2'\Theta\right\rangle\right| &\le \varepsilon\|W_0\|_2\mathbb{E}\|\mathcal{L}_z^*\mathbb{E}[\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta]\|_2\\
&\le \varepsilon\|W_0\|_2\left(\sup_{\mathbf{x},\mathbf{y}}\mathbb{E}^{1/2}|\delta_\varepsilon V_z^\varepsilon|^2\right)\mathbb{E}^{1/2}\|\tilde{\mathcal{L}}_z^*\tilde{\mathcal{L}}_z^*\Theta\|_2^2
\end{aligned}$$

which is $O(\varepsilon)$ by Lemma 2.

$\square$

Consider the test function $f^\varepsilon(z) = f(z) + f_1(z) + f_2(z) + f_3(z)$. We have

$$\begin{aligned}(49)\quad & \mathcal{A}^\varepsilon f^\varepsilon(z) \\ &= f'(z)\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta\rangle + f''(z)A_2^{(1)}(W_z^\varepsilon) + f'A_1^{(1)}(W_z^\varepsilon) + R_1(z) + R_2(z) + R_3(z).\end{aligned}$$

Set

$$R^\varepsilon(z) = R_1(z) + R_2(z) + R_3(z). \tag{50}$$

It follows from Propositions 3 and 5 that

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R^\varepsilon(z)| = 0.$$

Proposition 6.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\sup_{\|\psi\|_2=1} A_2^{(1)}(\psi) = 0.$$

Proof. We have

$$\begin{aligned} A_2^{(1)}(\psi) &= \int \psi(\mathbf{x},\mathbf{p})\mathcal{Q}_1(\Theta\otimes\Theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\psi(\mathbf{y},\mathbf{q})\,d\mathbf{x}d\mathbf{p}\,d\mathbf{y}d\mathbf{q} \\ &= \frac{1}{2}\int \psi(\mathbf{x},\mathbf{p})\widetilde{\mathcal{Q}}_1(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\psi(\mathbf{y},\mathbf{q})\,d\mathbf{x}d\mathbf{p}\,d\mathbf{y}d\mathbf{q}\end{aligned}$$

where $\widetilde{\mathcal{Q}}_1$ is defined by

$$\widetilde{\mathcal{Q}}_1(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q}) = \left[\mathcal{Q}_1(\Theta\otimes\Theta)(\mathbf{y},\mathbf{q},\mathbf{x},\mathbf{p}) + \mathcal{Q}_1(\Theta\otimes\Theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\right].$$

The symmetrized kernel has the following expressions

$$\begin{aligned}&\widetilde{\mathcal{Q}}_1(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q}) \\ &= \int_{-\infty}^{\infty} ds \int d\mathbf{p}'\check{\Phi}(s,\mathbf{p}')e^{i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{y})/\varepsilon^{2\alpha}}e^{-is\mathbf{p}\cdot\mathbf{p}'\varepsilon^{2-2\alpha}}\Big[e^{i\mathbf{p}'\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{p}'}{2\sqrt{k_1}}) \\ &\quad -e^{i\mathbf{p}'\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{p}'}{2\sqrt{k_2}})\Big]\Big[e^{i\mathbf{p}'\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{q}+\frac{\mathbf{p}'}{2\sqrt{k_1}}) - e^{i\mathbf{p}'\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{q}-\frac{\mathbf{p}'}{2\sqrt{k_2}})\Big] \\ &= 2\pi\int e^{i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{y})/\varepsilon^{2\alpha}}\Big[e^{i\mathbf{p}'\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}+\frac{\mathbf{p}'}{2\sqrt{k_1}}) - e^{i\mathbf{p}'\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{p}-\frac{\mathbf{p}'}{2\sqrt{k_2}})\Big] \\ &\quad\times\Big[e^{i\mathbf{p}'\cdot\tilde{\mathbf{x}}/\sqrt{k_1}}k_1\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{q}+\frac{\mathbf{p}'}{2\sqrt{k_1}}) - e^{i\mathbf{p}'\cdot\tilde{\mathbf{x}}/\sqrt{k_2}}k_2\nabla_{\mathbf{x}}\Theta(\mathbf{x},\mathbf{q}-\frac{\mathbf{p}'}{2\sqrt{k_2}})\Big]\Phi(\mathbf{p}\cdot\mathbf{p}'\varepsilon^{2-2\alpha},\mathbf{p}')d\mathbf{p}'.\end{aligned}$$

which, as the inverse Fourier transform tends to zero uniformly outside any neighborhood of $\mathbf{x}=\mathbf{y}$, because of Assumption 1, and stays uniformly bounded everywhere. Therefore the L^2-norm of $\widetilde{\mathcal{Q}}_1$ tends to zero and the proposition follows. □

Similar calculation leads to the following expression: For any real-valued, L^2-weakly convergent sequence $\psi^\varepsilon \to \psi$, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} A_1^{(1)}(\psi^\varepsilon) &= \lim_{\varepsilon\to 0}\int_0^\infty ds\int dwd\mathbf{q}d\mathbf{x}d\mathbf{p}\ \psi^\varepsilon(\mathbf{x},\mathbf{p})\Phi(w,\mathbf{q})e^{isw}e^{-is\mathbf{p}\cdot\mathbf{q}\varepsilon^{2-2\alpha}}\\
&\quad\times\Big\{e^{-i\frac{s|\mathbf{q}|^2}{2\sqrt{k_1}}\varepsilon^{2-2\alpha}}\Big[k_1k_2e^{i\mathbf{q}\cdot\mathbf{x}\varepsilon^{-2\alpha}(\frac{1}{\sqrt{k_1}}-\frac{1}{\sqrt{k_2}})}\Theta\Big(\mathbf{x},\mathbf{p}+\frac{1}{2}(\frac{1}{\sqrt{k_1}}+\frac{1}{\sqrt{k_2}})\mathbf{q}\Big)-k_1^2\Theta(\mathbf{x},\mathbf{p})\Big]\\
&\quad+e^{i\frac{s|\mathbf{q}|^2}{2\sqrt{k_2}}\varepsilon^{2-2\alpha}}\Big[k_1k_2e^{-i\mathbf{q}\cdot\mathbf{x}\varepsilon^{-2\alpha}(\frac{1}{\sqrt{k_1}}-\frac{1}{\sqrt{k_2}})}\Theta\Big(\mathbf{x},\mathbf{p}-\frac{1}{2}(\frac{1}{\sqrt{k_1}}+\frac{1}{\sqrt{k_2}})\mathbf{q}\Big)-k_2^2\Theta(\mathbf{x},\mathbf{p})\Big]\Big\}\\
&= k^2\lim_{\varepsilon\to 0}\int_0^\infty ds\int dwd\mathbf{q}d\mathbf{x}d\mathbf{p}\ \psi^\varepsilon(\mathbf{x},\mathbf{p})\Phi(w,\mathbf{q})e^{isw}e^{-is\mathbf{p}\cdot\mathbf{q}\varepsilon^{2-2\alpha}}\\
&\quad\times\Big\{e^{-i\frac{s|\mathbf{q}|^2}{2\sqrt{k}}\varepsilon^{2-2\alpha}}\Big[e^{i\mathbf{q}\cdot\mathbf{x}\beta/(2k^{1/2})}\Theta\Big(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{\sqrt{k}}\Big)-\Theta(\mathbf{x},\mathbf{p})\Big]\\
&\quad+e^{i\frac{s|\mathbf{q}|^2}{2\sqrt{k}}\varepsilon^{2-2\alpha}}\Big[e^{-i\mathbf{q}\cdot\mathbf{x}\beta/(2k^{1/2})}\Theta\Big(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{\sqrt{k}}\Big)-\Theta(\mathbf{x},\mathbf{p})\Big]\Big\}
\end{aligned}$$

where we have used (4). Note that the integrand is invariant under the change of variables: $s\to -s,\quad \mathbf{q}\to-\mathbf{q}$. Thus we can write

$$\begin{aligned}
&\lim_{\varepsilon\to 0} A_1^{(1)}(\psi^\varepsilon)\\
&= k^2\lim_{\varepsilon\to 0}\int_{-\infty}^\infty ds\int dwd\mathbf{q}d\mathbf{x}d\mathbf{p}\ \psi^\varepsilon(\mathbf{x},\mathbf{p})\Phi(w,\mathbf{q})e^{isw}e^{-is\mathbf{p}\cdot\mathbf{q}\varepsilon^{2-2\alpha}}e^{-i\frac{s|\mathbf{q}|^2}{2\sqrt{k}}\varepsilon^{2-2\alpha}}\\
&\quad\times\Big[e^{i\mathbf{q}\cdot\mathbf{x}\beta/(2k^{1/2})}\Theta\Big(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{\sqrt{k}}\Big)-\Theta(\mathbf{x},\mathbf{p})\Big]\\
&= 2\pi k^2\lim_{\varepsilon\to 0}\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\ \psi^\varepsilon(\mathbf{x},\mathbf{p})\Phi\big(\varepsilon^{2-2\alpha}(\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k}})\cdot\mathbf{q},\mathbf{q}\big)\Big[e^{i\mathbf{q}\cdot\mathbf{x}\beta/(2k^{1/2})}\Theta\big(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{\sqrt{k}}\big)-\Theta(\mathbf{x},\mathbf{p})\Big]
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\bar{A}_1(\psi) &\equiv \lim_{\varepsilon\to 0} A_1^{(1)}(\psi^\varepsilon)\\
&= \begin{cases} 2\pi k^2\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\ \psi(\mathbf{x},\mathbf{p})\Phi(0,\mathbf{q})\Big[e^{i\mathbf{q}\cdot\mathbf{x}\beta/(2k^{1/2})}\Theta\big(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{\sqrt{k}}\big)-\Theta(\mathbf{x},\mathbf{p})\Big], & \alpha\in(0,1)\\ 2\pi k^2\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\ \psi(\mathbf{x},\mathbf{p})\Phi\big((\mathbf{p}+\frac{\mathbf{q}}{2\sqrt{k}})\cdot\mathbf{q},\mathbf{q}\big)\Big[e^{i\mathbf{q}\cdot\mathbf{x}\beta/(2k^{1/2})}\Theta\big(\mathbf{x},\mathbf{p}+\frac{\mathbf{q}}{\sqrt{k}}\big)-\Theta(\mathbf{x},\mathbf{p})\Big], & \alpha=1\end{cases}.
\end{aligned}$$

Recall that

$$\begin{aligned}
M_z^\varepsilon(\Theta) &= f(z)+f_1(z)+f_2(z)+f_3(z)-\int_0^z f'(z)\left\langle W_z^\varepsilon,\mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta\right\rangle ds\\
&\quad-\int_0^z\Big[f''(s)A_2^{(1)}(W_s^\varepsilon)+f'(s)A_1^{(1)}(W_s^\varepsilon)\Big]\,ds-\int_0^z R^\varepsilon(s)\,ds \qquad (51)
\end{aligned}$$

is a martingale. The martingale property implies that for any finite sequence $0<z_1<z_2<z_3<\ldots<z_n\le z$, C^2-function f and bounded continuous function h with compact support, we have

$$\mathbb{E}\left\{h\left(\left\langle W_{z_1}^\varepsilon,\Theta\right\rangle,\left\langle W_{z_2}^\varepsilon,\Theta\right\rangle,\ldots,\left\langle W_{z_n}^\varepsilon,\Theta\right\rangle\right)\left[M_{z+s}^\varepsilon(\Theta)-M_z^\varepsilon(\Theta)\right]\right\} = 0, \qquad (52)$$
$$\forall s>0,\quad z_1\le z_2\le\cdots\le z_n\le z.$$

Let

$$\bar{\mathcal{A}}f(z)\equiv f'(z)\left[\left\langle W_z,\mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta\right\rangle+\bar{A}_1(W_z)\right].$$

Here and below, by slight abuse of notation, $f(z)$ and $f'(z)$ stand for $\phi(\langle W_z, \Theta\rangle)$ and $\phi'(\langle W_z, \Theta\rangle)$, respectively. In view of the results of Propositions 1,2, 3, 4, 5, 6 we see that

$$\begin{aligned}\mathbb{E}\left\{h\left(\left\langle W^\varepsilon_{z_1}, \Theta\right\rangle, \left\langle W^\varepsilon_{z_2}, \Theta\right\rangle, ..., \left\langle W^\varepsilon_{z_n}, \Theta\right\rangle\right)\left[f^\varepsilon(z) - \phi(\langle W^\varepsilon_z, \Theta\rangle)\right]\right\} &= 0,\\ \mathbb{E}\left\{h\left(\left\langle W^\varepsilon_{z_1}, \Theta\right\rangle, \left\langle W^\varepsilon_{z_2}, \Theta\right\rangle, ..., \left\langle W^\varepsilon_{z_n}, \Theta\right\rangle\right)\left[\mathcal{A}^\varepsilon f^\varepsilon(z) - \bar{\mathcal{A}}\phi(\langle W^\varepsilon_z, \Theta\rangle)\right]\right\} &= 0.\end{aligned}$$

With this and the tightness of W^ε_z we can pass to the limit $\varepsilon \to 0$ in (52), cf. [5], Chapter 4, Theorem 8.10. Consequently that the limiting process satisfies the martingale property that

$$\mathbb{E}\left\{h\left(\langle W_{z_1}, \Theta\rangle, \langle W_{z_2}, \Theta\rangle, ..., \langle W_{z_n}, \Theta\rangle\right)\left[M_{z+s}(\Theta) - M_z(\Theta)\right]\right\} = 0, \quad \forall s > 0.$$

where

$$M_z(\Theta) = f(z) - \int_0^z \bar{\mathcal{A}} f(s)\, ds.$$

Then it follows that

$$\mathbb{E}\left[M_{z+s}(\Theta) - M_z(\Theta) | W_u, u \le z\right] = 0, \quad \forall z, s > 0$$

which proves that $M_z(\Theta)$ is a martingale.

Choosing $\phi(r) = r$ and r^2 we see that

$$M^{(1)}_z(\Theta) = \langle W_z, \Theta\rangle - \int_0^z \left[\langle W_s, \mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta\rangle + \bar{A}_1(W_s)\right]\, ds$$

is a martingale with the null quadratic variation

$$\left[M^{(1)}(\Theta), M^{(1)}(\Theta)\right]_z = 0.$$

Thus

$$f(z) - \int_0^z \left\{f'(s)\left[\langle W_s, \mathbf{p}\cdot\nabla_{\mathbf{x}}\Theta\rangle + \bar{A}_1(W_s)\right]\right\} ds = f(0), \quad \forall z > 0.$$

Since $\langle W^\varepsilon_z, \Theta\rangle$ is uniformly bounded

$$|\langle W^\varepsilon_z, \Theta\rangle| \le \|W_0\|_2 \|\Theta\|_2$$

we have the convergence of the second moment

$$\lim_{\varepsilon\to 0} \mathbb{E}\left\{\langle W^\varepsilon_z, \Theta\rangle^2\right\} = \langle W_z, \Theta\rangle^2$$

and hence the convergence in probability.

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